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Solution of three-dimensional incompressible flow problems

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A method for solving quite general three-dimensional incompressible flow problems, in particular those described by the Navier–Stokes equations, is presented. The essence of the method is the expression of the velocity in terms of scalar and vector potentials, which are the three-dimensional generalizations of the two-dimensional stream function, and which ensure that the equation of continuity is satisfied automatically. Although the method is not new, a correct but simple and unambiguous procedure for using it has not been presented before.

Introduction

The time-dependent flow of an incompressible fluid in a three-dimensional domain R is governed by the equation of continuity

$$\nabla . \mathbf{u} = 0 \quad \text{in} \quad R \tag{1}$$

and the momentum equations, which in the case of the isothermal flow of a Newtonian fluid are the Navier–Stokes equations

$$\partial \mathbf{u}/\partial t + (\mathbf{u} . \nabla)\, \mathbf{u} = -\rho^{-1} \nabla p + \nu \Delta \mathbf{u} + \mathbf{F} \quad \text{in} \quad R, \tag{2}$$

together with boundary conditions $\mathbf{u} = \mathbf{v}$ on ∂R, the boundary of R, and initial conditions $\mathbf{u} = \mathbf{u}_0$ at time $t = 0$, where $\mathbf{u}$ and p denote velocity and pressure, respectively, ρ and ν denote density and kinematic viscosity, respectively, $\mathbf{F}$ denotes body forces which apply to the whole of a fluid element, t denotes time and Δ denotes the vector Laplacian, which is distinguished from the scalar Laplacian, denoted by ∇^2.

Two of the main difficulties inherent in determining the flow of an incompressible fluid are, first, that the momentum equations, whatever form they take, have to be solved subject to the continuity constraint (1) and second, that there is no evolution equation for the pressure. A method for overcoming these difficulties in two-dimensional flow problems, the vorticity/stream function method, is well known. The expression of the velocity in terms of a stream function ψ automatically ensures that the velocity field is solenoidal, so that the equation of continuity is satisfied, while the introduction of the vorticity ζ by cross-differentiation and subtraction of the momentum equations eliminates pressure as a dependent variable. Thus the difficulties of satisfaction of the continuity equation and the lack of an evolution equation for the pressure are eliminated in two-dimensional problems by replacing the primitive variables $\mathbf{u}$ and p by the derived variables ψ and ζ.

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It is often asserted that the vorticity/stream function method does not generalize to (non-axisymmetric) three-dimensional problems, but this is untrue. The three-dimensional generalization is the so-called vorticity/potential method. Here the expression of the velocity as the curl of a vector potential,

$$\mathbf{u} = \nabla \wedge \mathbf{A}, \tag{3}$$

ensures that the equation of continuity (1) is automatically satisfied, since the divergence of the curl of any vector field is identically zero. Similarly, the introduction of the vorticity

$$\boldsymbol{\omega} = \nabla \wedge \mathbf{u} \tag{4}$$

by taking the curl of the terms in the momentum equations eliminates pressure as a dependent variable, since the curl of the gradient of any scalar field is identically zero.

Although the vorticity/potential method is not new in hydrodynamics applications, there has been much confusion and unnecessary complication over its application, in particular over the boundary conditions appropriate to the vector potential. It has, for example, been asserted by Timman (1954) that, if the flow domain R is simply connected and the velocity $\mathbf{u}$ vanishes on the boundary ∂R of R, the vector potential $\mathbf{A}$ also vanishes on ∂R. This, as Moreau (1959) showed, is untrue. Similarly, Roache (1972) asserted that, if the velocity $\mathbf{u}$ vanishes on ∂R, the tangential components of the vector potential $\mathbf{A}$ vanish on ∂R, as does the normal derivative of the normal component of $\mathbf{A}$. This is true only if the domain R is simply connected and the boundary ∂R is planar. Again, Hirasaki & Hellums (1968) required the solution of a second-order partial differential equation in order to obtain the boundary conditions on the vector potential, which, although correct, is over-complex, and quite unnecessary. Later, Hirasaki & Hellums (1970) realized that a simplification in the boundary conditions is possible if a harmonic scalar potential ϕ is used as well as a vector potential $\mathbf{A}$:

$$\mathbf{u} = \nabla\phi + \nabla \wedge \mathbf{A}. \tag{5}$$

By choosing the particular boundary conditions for $\mathbf{A}$ that they did, however, they required ϕ to be (infinitely) many-valued in multiply connected flow domains, which is an increased and unnecessary complication.

The purpose of this paper is to give the correct set of boundary conditions on the scalar and vector potentials simply and concisely, and to show how to apply the vorticity/potential method unambiguously and correctly to solve quite general three-dimensional incompressible flow problems. This will be done as follows. First of all, certain results from potential theory will be presented. Although the basic results are not new, they are not well known and, in view of the errors made, in particular over the boundary conditions appropriate to the vector potential, they clearly need to be. Then, given the necessary theory, a procedure for solving arbitrary three-dimensional incompressible flow problems, in particular those described by the Navier–Stokes equations, will be presented. Finally, the advantages and disadvantages of using the vorticity/potential method to solve three-dimensional incompressible flow problems will be discussed.

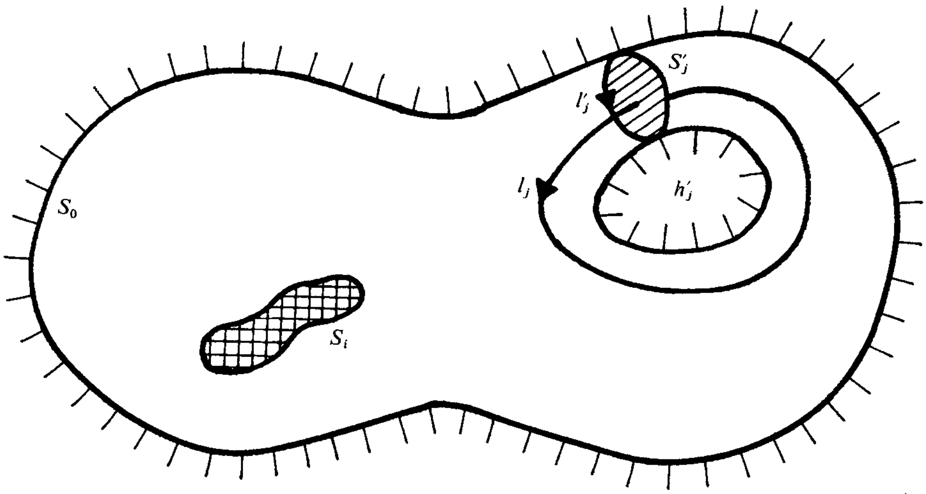

FIGURE 1. General three-dimensional domain R.

2. Potential theory

The aspect of potential theory which is of importance here is the expression of vector fields, in particular solenoidal vector fields, in terms of scalar and vector potential fields. The basic result that is presented is essentially a generalization of Helmholtz's theorem (see Aris 1962, p. 70), which states that an arbitrary, bounded, continuously differentiable vector field $\mathbf{u}$ which is defined throughout three-dimensional Euclidean space E_3 and vanishes at infinity can be expressed thus:

$$\mathbf{u} = \nabla\phi + \nabla \wedge \mathbf{A}, \tag{6}$$

where ϕ is the scalar potential and $\mathbf{A}$ is the vector potential of $\mathbf{u}$. The generalization of this result to arbitrary subspaces $R \subset E_3$, and to vector fields $\mathbf{u} \in L_2(R)$, the Hilbert space of Lebesgue square-integrable vector fields defined in R, i.e. those fields $\mathbf{u}$ for which

$$\iiint_R |\mathbf{u}|^2 dR < \infty,$$

is given in the theorem below.

Before the theorem is stated, however, it is necessary to describe the topology of a general three-dimensional domain $R \subset E_3$ (see figure 1). The boundary of R, denoted by ∂R, comprises (i) the outer boundary S_0 of R, part or all of which may be at infinity, and (ii) m surfaces S_i contained entirely within S_0 and so disconnected from S_0 and from each other. Thus one can put

$$\partial R = S_0 \cup S_1 \cup S_2 \cup \ldots \cup S_m.$$

The domain R may be multiply connected, and contain n holes of type h'_j, like the hole in a torus, so that R is, in fact, $(n+1)$ply connected. Such holes h'_j are characterized by closed contours l_j and by surfaces S'_j. The contours of type l_j are contained within R and completely encircle the hole h'_j; thus they cannot be continuously shrunk to a point without leaving R. The surfaces of type S'_j are bounded by closed contours l'_j on ∂R which cannot be continuously shrunk to a point without leaving ∂R. Note that the surfaces S_i can also contain holes of type h'_j.

The generalization of Helmholtz's theorem can now be given in the following theorem, which is due to Bykhovski & Smirnov (1960).

THEOREM. The Hilbert space $L_2(R)$ can be decomposed thus:

$$L_2(R) = G(R) \oplus U_1(R) \oplus U'(R) \oplus U_2(R) \oplus J(R), \tag{7}$$

where:

$G(R)$ is the closure of the space of infinitely differentiable vector fields of the form $\nabla\psi$ with ψ vanishing on ∂R;

$U_1(R)$ is the closure of the space of infinitely differentiable vector fields of the form

$$\sum_{i=1}^{m} \alpha_i \nabla h_i$$

such that the (scalar) Laplacian $\nabla^2 h_i$ of h_i is identically zero in R, m is the number of surfaces S_i, α_i is a constant, $h_i = \delta_{ik}$ (the Kronecker delta) on S_k and vanishes on S_0 and the two-period of ∇h_i,

$$\iint_S \nabla h_i \,.\, \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the unit normal to the element dS of S, vanishes identically unless S is a closed surface completely enclosing S_i;

$U'(R)$ is the closure of the space of infinitely differentiable vector fields of the form ∇h such that $\nabla^2 h \equiv 0$ in R and the two-period of ∇h over any closed surface vanishes identically;

$U_2(R)$ is the closure of the space of infinitely differentiable vector fields of the form

$$\sum_{j=1}^{n} \beta_j \nabla h^j$$

such that $\nabla^2 h^j \equiv 0$ in R, n is the number of holes h'_j, β_j is a constant, the one-period of ∇h^j,

$$\oint_{l_k} \nabla h^j \,.\, \mathbf{dl},$$

vanishes on all closed contours l_k except for contours l_j encircling the hole h'_j, where the one-period is unity (this implies that each h^j is many-valued), the normal derivative of h^j vanishes identically on ∂R and

$$\iint_{S'_k} \nabla h^j \,.\, \mathbf{n}\, dS$$

vanishes identically except on the surface S'_j, where $\mathbf{n}$ is the unit normal to the element dS of S'_k;

$J(R)$ is the closure of the space of infinitely differentiable vector fields of the form $\nabla \wedge \mathbf{V}$ such that $\nabla \,.\, \mathbf{V} \equiv 0$ in R, the tangential components of $\mathbf{V}$ vanish identically on ∂R and

$$\iint_{S'_k} \nabla \wedge \mathbf{V} \,.\, \mathbf{n}\, dS$$

vanishes identically for all S'_k, where $\mathbf{n}$ is the unit normal to the element dS of S'_k.

The symbol $\oplus$ denotes an orthogonal sum, which means that vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ from different subspaces of $L_2(R)$ are mutually orthogonal in the sense that their inner product

$$\iiint_R \mathbf{u}_1 \cdot \mathbf{u}_2 \, dR$$

vanishes identically.

COROLLARY 1. Any vector field $\mathbf{u} \in L_2(R)$ can be approximated to within an arbitrarily small error thus:

$$\mathbf{u} = \nabla\psi + \sum_{i=1}^{m} \alpha_i \nabla h_i + \nabla h + \sum_{j=1}^{n} \beta_j \nabla h^j + \nabla \wedge \mathbf{V}, \tag{8}$$

with the various terms defined by the conditions given in the theorem.

COROLLARY 2. Any vector field $\mathbf{u} \in U_1(R) \oplus U'(R) \oplus U_2(R)$ can be approximated to within an arbitrarily small error thus:

$$\mathbf{u} = \nabla\phi, \tag{9}$$

where $$\nabla^2\phi \equiv 0 \quad \text{in} \quad R$$

and $$\nabla\phi \, . \, \mathbf{n} = \mathbf{u} \, . \, \mathbf{n} \quad \text{on} \quad \partial R.$$

COROLLARY 3. Any vector field $\mathbf{u} \in U'(R) \oplus U_2(R) \oplus J(R)$ can be approximated to within an arbitrarily small error thus:

$$\mathbf{u} = \nabla \wedge \mathbf{A}, \tag{10}$$

where $$\nabla \, . \, \mathbf{A} \equiv 0 \quad \text{in} \quad R,$$

$$\mathbf{A} \, . \, \mathbf{n} \equiv 0 \quad \text{on} \quad \partial R$$

and $$\iint_{S'_k} \mathbf{A} \, . \, \mathbf{n} \, dS \equiv 0 \quad \text{for all} \quad S'_k.$$

Corollary 1 is almost obvious. Corollary 2 follows from corollary 1 by combining

$$\sum_{i=1}^{m} \alpha_i h_i, \quad h, \quad \sum_{j=1}^{n} \beta_j h^j$$

to form ϕ. Corollary 3 is due to Bykhovski & Smirnov (1960, theorem 4.3, p. 32).

In what follows, it will be assumed not only that $\mathbf{u}$ belongs to $L_2(R)$, but also that it is infinitely differentiable in R. Although this further assumption is not necessary, because all results to be given hold to within an arbitrarily small error for all vector fields in $L_2(R)$, it will simplify matters considerably; for example, expression (8) holds exactly for infinitely differentiable vector fields in $L_2(R)$.

Expression (8) is rather too specific a representation of an arbitrary vector field for the purposes of determining the flow of an incompressible fluid. A less specific expression is required. It follows from corollaries 2 and 3 that the $U'(R)$ and $U_2(R)$ components of any vector field can be expressed either in the form ∇H or in the form $\nabla \wedge \mathbf{H}$, and hence that any solenoidal vector field can be represented thus:

$$\mathbf{u} = \nabla\phi + \nabla \wedge \mathbf{A}, \tag{11}$$

where $\nabla^2\phi \equiv 0$ in R and $\nabla \, . \, \mathbf{A} \equiv 0$ in R (note that ϕ and $\mathbf{A}$ in (11) are not necessarily the same as ϕ and $\mathbf{A}$ in (9) and (10), respectively). The use of the less specific representation (11) means, however, that a degree of arbitrariness is involved. Thus one can do any of the following.

(i) Express both the $U'(R)$ and the $U_2(R)$ components of $\mathbf{u}$ as the curls of vector potentials.

(ii) Express both the $U'(R)$ and the $U_2(R)$ components of $\mathbf{u}$ as the gradients of scalar potentials.

(iii) Express the $U'(R)$ component of $\mathbf{u}$ as the gradient of a scalar potential and the $U_2(R)$ component of $\mathbf{u}$ as the curl of a vector potential.

(iv) Express the $U'(R)$ component of $\mathbf{u}$ as the curl of a vector potential and the $U_2(R)$ component of $\mathbf{u}$ as the gradient of a scalar potential.

It will be noted, however, that three of these four possibilities are less convenient than the fourth:

(*a*) Possibilities (ii) and (iv) involve the expression of the $U_2(R)$ component of $\mathbf{u}$ as the gradient of an (infinitely) many-valued scalar potential. It is clearly more convenient if all variables are single-valued; for this reason, possibilities (ii) and (iv) may be rejected.

(*b*) The $U_1(R)$ and $U'(R)$ components of the solenoidal vector field $\mathbf{u}$ are together responsible for all of the flux of $\mathbf{u}$ through the boundary ∂R of the domain R. If both of these components are combined into a single component of the form of the gradient of a scalar potential then, as corollary 2 shows, this single component will be defined by a Neumann problem. For this reason, possibilities (i) and (iv) may be rejected, because they involve the expression of the $U'(R)$ component of $\mathbf{u}$ as the curl of a vector potential.

Thus it can be concluded that the most convenient form of the representation (11) is given by possibility (iii), that is

$$\mathbf{u} = \nabla\phi + \nabla \wedge \mathbf{A}, \tag{12}$$

where
$$\left.\begin{aligned} &\nabla\phi \in U_1(R) \oplus U'(R), \quad \nabla^2\phi \equiv 0 \text{ in } R \\ &\nabla \wedge \mathbf{A} \in U_2(R) \oplus J(R), \quad \nabla . \mathbf{A} \equiv 0 \text{ in } R. \end{aligned}\right\} \tag{13}$$
and

The boundary conditions on ϕ and $\mathbf{A}$, which follow directly from corollaries 2 and 3, are

$$\frac{1}{s_n}\frac{\partial\phi}{\partial\xi_n} = v_n, \tag{14}$$

$$\left.\begin{aligned} A_n &\equiv 0, \\ -\frac{1}{s_n s_{t_2}}\frac{\partial}{\partial\xi_n}(s_{t_2}A_{t_2}) &= v_{t_1} - \frac{1}{s_{t_1}}\frac{\partial\phi}{\partial\xi_{t_1}}, \\ +\frac{1}{s_n s_{t_1}}\frac{\partial}{\partial\xi_n}(s_{t_1}A_{t_1}) &= v_{t_2} - \frac{1}{s_{t_2}}\frac{\partial\phi}{\partial\xi_{t_2}}, \end{aligned}\right\} \tag{15}$$

where $\mathbf{u} = \mathbf{v}$ on ∂R; s_n, s_{t_1} and s_{t_2} are scale factors; n, t_1 and t_2 denote the normal and two tangential directions to ∂R; and ξ_n, ξ_{t_1} and ξ_{t_2} denote the normal and two tangential co-ordinates relative to ∂R (see appendix). To conclude this section, note that, if the domain R is simply connected, the boundary conditions on $\mathbf{A}$ simplify to

$$\left.\begin{aligned} A_{t_1} \equiv 0, \quad A_{t_2} \equiv 0, \\ \partial(s_{t_1}s_{t_2}A_n)/\partial\xi_n \equiv 0 \end{aligned}\right\} \tag{16}$$

in the same notation. The reason for this is that if the domain R is simply connected $\nabla \wedge \mathbf{A}$ can have no $U_2(R)$ component, which means that $\nabla \wedge \mathbf{A} \in J(R)$. Use of the conditions on vector fields in $J(R)$ given in the theorem then leads to the zero-tangential-component conditions on $\mathbf{A}$, while the fact that $\mathbf{A}$ is solenoidal leads to the condition on the normal derivative of the normal component of $\mathbf{A}$.

3. Solution of the Navier–Stokes equations

From the theory presented in the previous section, it is possible to give explicitly an unambiguous procedure for solving three-dimensional incompressible flow problems. In this section, the analytical formulation of a method for solving the three-dimensional time-dependent Navier–Stokes equations will be presented. Clearly, the method can be extended in a straightforward manner to deal with other incompressible flow problems, such as solution of the steady-state Navier–Stokes equations, or solution of the equations of motion of incompressible non-Newtonian fluids.

Given. (i) The geometry of the flow domain R, a subspace of E_3, with boundary ∂R.

(ii) Boundary conditions on the velocity field $\mathbf{u}$:

$$\mathbf{u} = \mathbf{v} \quad \text{on} \quad \partial R,$$

where $\mathbf{v}$ is such that

$$\iint_{\partial R} \mathbf{v} . \mathbf{n} \, dS = 0.$$

(iii) Initial conditions on the velocity field $\mathbf{u}$:

$$\mathbf{u} = \mathbf{u}_0 \quad \text{at time} \quad t = 0,$$

where $\mathbf{u}_0$ is such that $\nabla . \mathbf{u}_0 = 0$ and $\mathbf{u} = \mathbf{v}$ on ∂R at time $t = 0$.

(iv) The body forces $\mathbf{F}$, which act on the whole of a fluid element, at all points in R for all times $t \geqslant 0$.

(v) The fluid density ρ and kinematic viscosity ν.

Solve. The three-dimensional time-dependent Navier–Stokes equations (2) for the velocity field $\mathbf{u}$ and pressure field p, subject to the constraint of the continuity equation (1).

Procedure. (i) Determine the appropriate potential representation of the velocity field:

(*a*) In general, the velocity field $\mathbf{u}$ can be expressed thus:

$$\mathbf{u} = \nabla\phi + \nabla \wedge \mathbf{A}, \tag{17}$$

where ϕ is a scalar potential field satisfying the Laplace equation

$$\nabla^2\phi \equiv 0 \quad \text{in} \quad R \tag{18}$$

and $\mathbf{A}$ is a solenoidal (i.e. $\nabla . \mathbf{A} \equiv 0$ in R) vector potential field satisfying the Poisson equation

$$\Delta\mathbf{A} = -\nabla \wedge \nabla \wedge \mathbf{A} = -\boldsymbol{\omega} \quad \text{in} \quad R, \tag{19}$$

where $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$ is the vorticity field.

(*b*) The boundary conditions on ϕ and $\mathbf{A}$ depend on the nature of the flow at the boundary ∂R of the flow domain and on the topological properties of the flow domain, respectively.

(b 1) If there is flow through any part of ∂R, or ∂R is itself moving, then the boundary condition on ϕ is

$$s_n^{-1}\,\partial\phi/\partial\xi_n = v_n \quad \text{on} \quad \partial R, \tag{20}$$

where v_n denotes the normal component of $\mathbf{v}$, s_n is a scale factor and ξ_n is a co-ordinate normal to ∂R (see appendix). If there is no flow through any part of ∂R, and ∂R is not moving at any point, then

$$s_n^{-1}\,\partial\phi/\partial\xi_n \equiv 0 \quad \text{on} \quad \partial R, \tag{21}$$

so that

$$\nabla\phi \equiv 0 \quad \text{in } R \text{ and on } \partial R. \tag{22}$$

(b 2) If the domain R is simply connected, the boundary conditions on $\mathbf{A}$ are

$$\left.\begin{array}{c} A_{t_1} \equiv 0, \quad A_{t_2} \equiv 0 \\ \partial(s_{t_1}s_{t_2}A_n)/\partial\xi_n \equiv 0 \end{array}\right\} \quad \text{on} \quad \partial R. \tag{23}$$

If the domain R is multiply connected, the boundary conditions on $\mathbf{A}$ are

$$\left.\begin{array}{c} A_n \equiv 0 \\ -\dfrac{1}{s_n s_{t_2}}\dfrac{\partial}{\partial\xi_n}(s_{t_2}A_{t_2}) = v_{t_1} - \dfrac{1}{s_{t_1}}\dfrac{\partial\phi}{\partial\xi_{t_1}} \\ +\dfrac{1}{s_n s_{t_1}}\dfrac{\partial}{\partial\xi_n}(s_{t_1}A_{t_1}) = v_{t_2} - \dfrac{1}{s_{t_2}}\dfrac{\partial\phi}{\partial\xi_{t_2}} \end{array}\right\} \quad \text{on} \quad \partial R, \tag{24}$$

where the subscripts n, t_1 and t_2 denote the normal and two tangential directions to ∂R, ξ denotes a co-ordinate, and s_n, s_{t_1} and s_{t_2} are scale factors (see appendix).

(c) The initial conditions on ϕ are given by (18) and (20) or (21) from the initial conditions on $\mathbf{u}$. The initial conditions on $\mathbf{A}$ are given by (19) and (23) or (24) from the initial conditions on $\mathbf{u}$.

(ii) Obtain the evolution equation for the vorticity:

(a) Taking the curl of each of the terms in the Navier–Stokes equations (2) gives the vorticity transport equation:

$$\partial\boldsymbol{\omega}/\partial t + (\mathbf{u}\,.\,\nabla)\,\boldsymbol{\omega} - (\boldsymbol{\omega}\,.\,\nabla)\,\mathbf{u} = \nu\Delta\boldsymbol{\omega} + \nabla\wedge\mathbf{F}, \tag{25}$$

where $\boldsymbol{\omega} = \nabla\wedge\mathbf{u}$ is the vorticity. Substituting for $\mathbf{u}$ from (17) gives

$$\partial\boldsymbol{\omega}/\partial t + ((\nabla\phi + \nabla\wedge\mathbf{A})\,.\,\nabla)\,\boldsymbol{\omega} - (\boldsymbol{\omega}\,.\,\nabla)\,(\nabla\phi + \nabla\wedge\mathbf{A}) = \nu\Delta\boldsymbol{\omega} + \nabla\wedge\mathbf{F}. \tag{26}$$

(b) The initial conditions on $\boldsymbol{\omega}$ are given from the definition of $\boldsymbol{\omega}$ by the initial conditions on $\mathbf{u}$. The boundary conditions on $\boldsymbol{\omega}$, which are not all known *a priori* for times $t > 0$ (so that an iterative scheme must in general be used to determine them), are given from the definition of $\boldsymbol{\omega}$ by a knowledge of $\mathbf{u}$ in the neighbourhood of and on the boundary ∂R.

(iii) Implement the solution:

Equations (18), (19), (20) or (21), (23) or (24) and (26) form a complete set defining ϕ, $\mathbf{A}$ and $\boldsymbol{\omega}$ at all points in the flow domain R and on its boundary ∂R, for all times $t \geqslant 0$. In general, the implementation of the solution will be of a numerical nature, and will involve discretization of the flow field and time, and replacement of the equations and boundary and initial conditions by numerical analogues. (The manner in which the discretization is performed and the choice of numerical analogues depend on the particular problem to be solved.) The fields ϕ, $\mathbf{A}$ and $\boldsymbol{\omega}$ can then be determined approximately at all points in the discretized flow domain R and on its boundary ∂R, for all discretized times $t \geqslant 0$. Finally, the fields $\mathbf{u}$ and p can be obtained (approximately) by substitution back into (17) and (2), respectively.

4. The vorticity/potential method: discussion and conclusions

The procedure given in the preceding section for solving, in particular, the three-dimensional time-dependent Navier–Stokes equations for incompressible flows is based on the quite general theory of § 2. Although the basic theory presented there is not new, the results relevant to hydrodynamics problems are not well known and are generally given either incorrectly or in an over-complex (and hence impractical) form. Here the relevant results are given both correctly and in a simple form. The results are applicable to quite general incompressible flow problems; failing cases, i.e. non-$L_2(R)$ velocity fields, are unlikely to be encountered physically, because infinite kinetic energy would be implied. Thus the obvious advantages of the vorticity/potential method are that:

(i) it is applicable to quite general incompressible flow problems;

(ii) it overcomes the difficulties over imposition of the continuity constraint and lack of a pressure evolution equation, inherent in incompressible flow problems;

(iii) it is straightforward to use in practice.

Disadvantages associated with the method, on the other hand, are that:

(i) use of the vorticity/potential method involves seven dependent variables (three vorticity and three vector-potential components plus a scalar potential), whereas use of the primitive variables involves only four (three velocity components plus pressure);

(ii) unless a Cartesian co-ordinate system is used, the equations involved in the vorticity/potential method are much more complex than the equations associated with the primitive variables;

(iii) the primitive variables have a direct physical meaning, as does vorticity, but the potentials do not.

Clearly, for a given problem, certain advantages and/or disadvantages will weigh more heavily than others. But, for a variety of problems, such as three-dimensional natural convection (see Aziz & Hellums 1967; Mallinson & de Vahl Davis 1973), thermal convection in confined porous media (see Holst & Aziz 1972), flow between parallel flat plates (see Hirasaki 1967), and uniform and linear-shear flow past a sphere (see Richardson 1976), the method has been used successfully. Its use in other three-dimensional flow problems, whether time-dependent or steady, Newtonian or non-Newtonian, can, therefore, be confidently recommended.

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Appendix

It is well known that any piecewise smooth surface ∂R bounding a domain $R \subset E_3$ can be described locally by a mutually orthogonal curvilinear co-ordinate system $(\xi_n, \xi_{t_1}, \xi_{t_2})$ nearly everywhere, where n denotes the normal and t_1 and t_2 the two tangential directions to ∂R (see figure 2). By convention, this co-ordinate system $(\xi_n, \xi_{t_1}, \xi_{t_2})$ is taken to be right-handed, and ξ_n is taken to denote the outer normal.

Let $\mathbf{r} = \mathbf{r}(\xi_n, \xi_{t_1}, \xi_{t_2})$ be the position vector of a point on ∂R. A unit tangent in the ξ_i direction, where i denotes n, t_1 or t_2, is

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial \xi_i} \bigg/ \left| \frac{\partial \mathbf{r}}{\partial \xi_i} \right|, \tag{A 1}$$

so that

$$\partial \mathbf{r}/\partial \xi_i = s_i \, \mathbf{e}_i, \tag{A 2}$$

where

$$s_i = |\partial \mathbf{r}/\partial \xi_i|. \tag{A 3}$$

The quantities s_i are called scale factors. Expressions for the gradient, divergence and curl operators in terms of the orthogonal curvilinear co-ordinates are then (see, for example, Spiegel 1959, p. 137)

$$\nabla \psi = \frac{1}{s_n} \frac{\partial \psi}{\partial \xi_n} \mathbf{e}_n + \frac{1}{s_{t_1}} \frac{\partial \psi}{\partial \xi_{t_1}} \mathbf{e}_{t_1} + \frac{1}{s_{t_2}} \frac{\partial \psi}{\partial \xi_{t_2}} \mathbf{e}_{t_2}, \tag{A 4}$$

$$\nabla . \mathbf{V} = \frac{1}{s_n s_{t_1} s_{t_2}} \left\{ \frac{\partial}{\partial \xi_n} (s_{t_1} s_{t_2} V_n) + \frac{\partial}{\partial \xi_{t_1}} (s_{t_2} s_n V_{t_1}) + \frac{\partial}{\partial \xi_{t_2}} (s_n s_{t_1} V_{t_2}) \right\}, \tag{A 5}$$

$$\begin{aligned} \nabla \wedge \mathbf{V} = & \frac{1}{s_{t_1} s_{t_2}} \left\{ \frac{\partial}{\partial \xi_{t_1}} (s_{t_2} V_{t_2}) - \frac{\partial}{\partial \xi_{t_2}} (s_{t_1} V_{t_1}) \right\} \mathbf{e}_n \\ & + \frac{1}{s_{t_2} s_n} \left\{ \frac{\partial}{\partial \xi_{t_2}} (s_n V_n) - \frac{\partial}{\partial \xi_n} (s_{t_2} V_{t_2}) \right\} \mathbf{e}_{t_1} \\ & + \frac{1}{s_n s_{t_1}} \left\{ \frac{\partial}{\partial \xi_n} (s_{t_1} V_{t_1}) - \frac{\partial}{\partial \xi_{t_1}} (s_n V_n) \right\} \mathbf{e}_{t_2}. \end{aligned} \tag{A 6}$$

By convention, one generally writes $\mathbf{n}$ for $\mathbf{e}_n$.

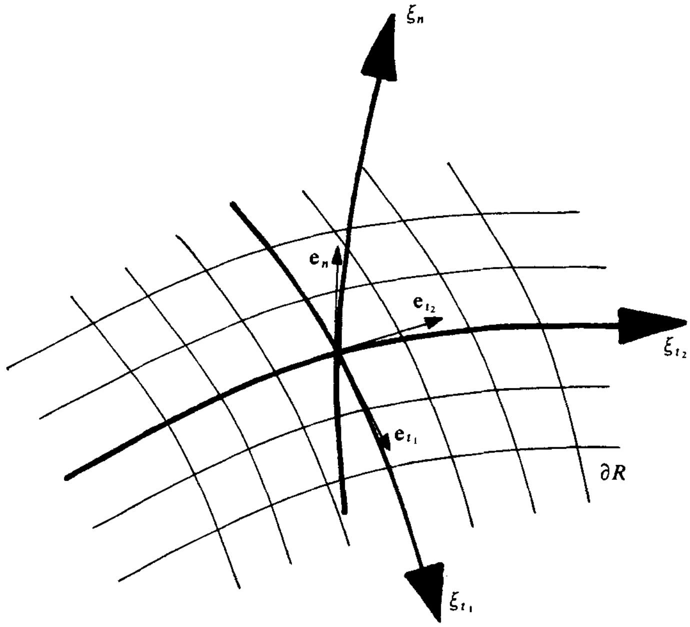

FIGURE 2. Co-ordinate system on the boundary ∂R of R.

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